DTMC II

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Calculation of stationary distribution finite chain

- brute-force (when there is limiting distribution)

is simply to take the transition matrix to a high power and then extract out any row

- Solving global balance: via eigendecomposition

Note that the equation $\pi \cdot P = (\lambda=1)\pi$ implies that the row vector π (if π is a column vector use π^T as a row vector) is a left eigenvector of P with eigenvalue equal to 1 (Recall $xA=\lambda x$ where x is a row vector is definition of a left eigenvector, as opposed to the more standard right eigenvector $Ax=\lambda x$)

- Solving global balance : via a system of linear equations solution

solve the system of linear equations $\pi \cdot P = \pi$, $\Sigma_i \pi_i = 1$ (assume π is row vector)

These equations are known as the global balance equations, and this approach is introduced in Discrete Markov Chains: Finding the Stationary Distribution via solution of the global balance equations

Calculation of limiting Distribution (P^n): finite chain-generating function method (Kobayashi P429) $\pi_j(n+1)$, $\boldsymbol{\pi}^{\mathrm{T}}$

- We have $p_j(n+1) = \sum_{i\in S} p_i(n)P_{ij}(n) \qquad j \in S$
- Or in Matrix form $\boldsymbol{p}^{\mathrm{T}}(n+1) = \boldsymbol{p}^{\mathrm{T}}(n)\boldsymbol{P}$ where $\boldsymbol{p}^{\mathrm{T}}(n)$ is a row vector of dimension $|S|$
- If MC is homogeneous we find $\boldsymbol{p}^{\mathrm{T}}(n+1) = \boldsymbol{p}^{\mathrm{T}}(n-1)\boldsymbol{P}\boldsymbol{P} = \ldots = \boldsymbol{p}^{\mathrm{T}}(0)\boldsymbol{P}^{\mathrm{n}}$
- let $\boldsymbol{g}(z)$ denote the generating function of the vector sequence $\{\boldsymbol{p}(n); n = 0, 1, 2, \ldots\}$, i.e.,

$$\boldsymbol{g}(z) = \sum_{n=0}^{\infty} \boldsymbol{p}(n) z^n$$

Calculation of limiting Distribution (P^n): finite chain-generating function method

$$g(z) = \sum_{n=0}^{\infty} p(n) z^n$$

- From $\quad p^{T}(n+1) = \ p^{T}(n) P \quad$ for all n=0,1,…

- and $\quad \sum_{n=0}^{\infty} p^{T}(n+1) z^n = \sum_{n=0}^{\infty} p^{T}(n) z^n P \ = g^{T}(z) P$

- We have

$$g^{T}(z)P = \sum_{n=0}^{\infty} p^{T}(n+1) z^n = z^{-1} \sum_{n=0}^{\infty} p^{T}(n+1) z^{n+1} = z^{-1} \sum_{n=0}^{\infty} p^{T}(n+1) z^{n+1} + z^{-1} p^{T}(0) z^0 - z^{-1} p^{T}(0) z^0$$

$$= z^{-1} g^{T}(z) - z^{-1} p^{T}(0)$$

- And thus

$$g^{T}(z) = p^{T}(0)[I - Pz]^{-1}$$

- where ***I*** is the *M* × *M* identity matrix, where *M* = |*S*|.

Calculation of limiting Distribution (P^n): finite chain- generating function method example

- Assume

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

- Then

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{16} & \frac{9}{16} & \frac{3}{8} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}.$$

$$P^3 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{16} & \frac{9}{16} & \frac{3}{8} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{16} & \frac{9}{16} & \frac{3}{8} \\ \frac{9}{64} & \frac{25}{64} & \frac{15}{32} \\ \frac{3}{32} & \frac{15}{32} & \frac{7}{16} \end{bmatrix}. \qquad \lim_{n\to\infty} P^n = P^\infty = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{bmatrix}.$$

$$\lim_{n\to\infty} \boldsymbol{p}^{\mathrm{T}}(n) = \boldsymbol{p}^{\mathrm{T}}(0)\boldsymbol{P}^\infty = (0\ 1\ 0)\begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{bmatrix} = \left(\frac{1}{9}\ \frac{4}{9}\ \frac{4}{9}\right)$$

Calculation of limiting Distribution (P^n): finite chain- generating function method example

- the inverse of a matrix ***A*** can be expressed as

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det|A|},$$

- where adj (***A***) is the *adjugate* (called *classical adjoint*) matrix of ***A*** .
- Applying this result to

$$\boldsymbol{g}^{\mathrm{T}}(z) = \boldsymbol{p}^{\mathrm{T}}(0)[\boldsymbol{I} - \boldsymbol{P}z]^{-1}$$

- we have

$$[\boldsymbol{I} - \boldsymbol{P}z]^{-1} = \frac{\boldsymbol{B}(z)}{\Delta(z)}$$

Calculation of limiting Distribution (P^n): finite chain- generating function method example

$$I - Pz = \begin{bmatrix} 1 & -z & 0 \\ -\frac{z}{4} & 1-\frac{z}{4} & -\frac{z}{2} \\ 0 & -\frac{z}{2} & 1-\frac{z}{2} \end{bmatrix}$$

$$\begin{vmatrix} 1-\frac{z}{4} & -\frac{z}{2} \\ -\frac{z}{2} & 1-\frac{z}{2} \end{vmatrix} = \left(1-\frac{z}{4}\right)\left(1-\frac{z}{2}\right) - \frac{z^2}{4} = 1 - \frac{3z}{4} - \frac{z^2}{8}$$

.... Calculate other det.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\operatorname{adj}(\mathbf{A}) = \mathbf{C}^{\mathsf{T}} = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

$$\begin{vmatrix} a_{im} & a_{in} \\ a_{jm} & a_{jn} \end{vmatrix} = \det \begin{bmatrix} a_{im} & a_{in} \\ a_{jm} & a_{jn} \end{bmatrix}$$

Calculation of limiting Distribution (P^n): finite chain- generating function method example

- Where

$$\boldsymbol{B}(z) = adj\ (\boldsymbol{I} - \boldsymbol{P}z)] = \begin{bmatrix} 1 - \frac{3z}{4} - \frac{z^2}{8} & z\left(1 - \frac{z}{2}\right) & \frac{z^2}{2} \\ \frac{z}{4}\left(1 - \frac{z}{2}\right) & 1 - \frac{z}{2} & \frac{z}{2} \\ \frac{z^2}{8} & \frac{z}{2} & 1 - \frac{z}{4} - \frac{z^2}{4} \end{bmatrix}$$

- And

$$\Delta(z) \triangleq \det[\boldsymbol{I} - \boldsymbol{P}z] = (1 - z)(1 + \frac{z}{4} - \frac{z^2}{8})$$

- Hence

$$[\boldsymbol{I} - \boldsymbol{P}z]^{-1} = \frac{1}{\Delta(z)} \begin{bmatrix} 1 - \frac{3z}{4} - \frac{z^2}{8} & z\left(1 - \frac{z}{2}\right) & \frac{z^2}{2} \\ \frac{z}{4}\left(1 - \frac{z}{2}\right) & 1 - \frac{z}{2} & \frac{z}{2} \\ \frac{z^2}{8} & \frac{z}{2} & 1 - \frac{z}{4} - \frac{z^2}{4} \end{bmatrix}$$

Calculation of limiting Distribution (P^n): finite chain- generating function method example

Then by substituting the obtained expression and $\boldsymbol{p}^{\mathrm{T}}(0) = (p_0(0) \quad p_1(0) \quad p_2(0))$

- Into

$$\boldsymbol{g}^{\mathrm{T}}(z) = \boldsymbol{p}^{\mathrm{T}}(0)[\boldsymbol{I} - \boldsymbol{P}z]^{-1} = (p_0(0) \quad p_1(0) \quad p_2(0)) \frac{1}{\Delta(z)} \begin{bmatrix} 1 - \frac{3z}{4} - \frac{z^2}{8} & z\left(1 - \frac{z}{2}\right) & \frac{z^2}{2} \\ \frac{z}{4}\left(1 - \frac{z}{2}\right) & 1 - \frac{z}{2} & \frac{z}{2} \\ \frac{z^2}{8} & \frac{z}{2} & 1 - \frac{z}{4} - \frac{z^2}{4} \end{bmatrix}$$

- We obtain

$$\boldsymbol{g}^{\mathrm{T}}(z) = (g_0(z) \quad g_1(z) \quad g_2(z))$$

where

$$g_0(z) = \frac{1}{\Delta(z)}\left[p_0(0)\left(1 - \frac{3z}{4} - \frac{z^2}{8}\right) + p_1(0)\frac{z}{4}\left(1 - \frac{z}{2}\right) + p_2(0)\frac{z^2}{2}\right]$$

$$g_1(z) = \frac{1}{\Delta(z)}\left[p_0(0)z\left(1 - \frac{z}{2}\right) + p_1(0)\left(1 - \frac{z}{2}\right) + p_2(0)\frac{z}{2}\right]$$

$$g_2(z) = \frac{1}{\Delta(z)}\left[p_0(0)\frac{z^2}{8} + p_1(0)\frac{z}{2} + p_2(0)\left(1 - \frac{z}{4} - \frac{z^2}{4}\right)\right]$$

Calculation of limiting Distribution (P^n): finite chain- generating function method example

- Inverting the PGF , we find $\{p_i(n); i = 0, 1, 2, n = 0, 1, 2, \ldots\}$.
- The limiting probabilities $\lim_{n\to\infty} p_i(n)$ can be obtained, however, without going through the inversion of the PGF. By applying the **final value theorem**,(see next 2 slides) we have

$$\lim_{n\to\infty} p_0(n) = \lim_{z\to 1}(1-z)g_0(z) = \frac{1}{9}$$

$$\lim_{n\to\infty} p_1(n) = \lim_{z\to 1}(1-z)g_1(z) = \frac{4}{9}$$

$$\lim_{n\to\infty} p_2(n) = \lim_{z\to 1}(1-z)g_2(z) = \frac{4}{9}$$

Calculation of limiting Distribution (P^n): finite chain- generating function method useful note

- Problem 9.7 (Kobayashi P236):
- **Generating function of a sequence.** Let *F(z)* be the generating function of a sequence or vector $\{ f_k ; k = 0, 1, 2, . . .\}$ defined by
- Find $\{ f_k ; k = 0, 1, 2, . . .\}$ for the following *F(z)*:

$$F(z) = \sum_{k=0}^{\infty} f_k z^k$$

$$(a) \qquad F(z) = \frac{1}{1 - \alpha z};$$

$$(b) \qquad F(z) = \frac{1}{(1-\alpha z)^2};$$

$$(c) \qquad F(z) = \frac{\alpha z}{(1-\alpha z)^2};$$

Calculation of limiting Distribution (P^n): finite chain- generating function method useful note

- **Final value theorem. (** Problem 9.12.)
- Refer to previous slide (problem 9.7) and show that

$$\lim_{z\to 1}(1-z)F(z) = \lim_{k\to\infty} f_k$$

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method (Kobayashi 433)

- An alternative method to evaluate the state probability vector ***p****(n)* is to use the eigenvalues and eigenvectors of the TPM ***P***. (**spectral expansion** method, since the set of eigenvalues of a matrix is also called its **spectrum**.)
- This method is similar to the generalized Fourier series expansion or Karhunen– Loève expansion method.
- In this case, however, ***P*** is not a symmetric matrix.

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method (Papoulis 4th ed page 706)

- Let λ_i be the ith **eigenvalue** and $\boldsymbol{u}_i$ be the associated **right-eigenvector** of the Markov TPM

$\boldsymbol{P}\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i$, $i \in S = \{0, 1, 2, \ldots, M-1\}$,

where $M = |S|$ is the number of states and $\boldsymbol{u}_i$ is a column vector, making its transpose $\boldsymbol{u}_i^T$ a row vector.

- We assume that all eigenvalues are distinct; i.e., there is no multiplicity of any of the eigenvalues.
- Let us form an $M \times M$ matrix $\boldsymbol{U}$ by $\boldsymbol{U} = [\boldsymbol{u}_0 \;\; \boldsymbol{u}_1 \;\; \cdots \;\; \boldsymbol{u}_{M-1}]$ and a diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_0 & 0 & & \dots & 0 \\ 0 & \lambda_1 & & \dots & 0 \\ & & \dots & & \\ 0 & 0 & \dots & & \lambda_{M-1} \end{bmatrix}$$

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method (Papoulis)

- The TPM can be expanded as $\boldsymbol{P} = \boldsymbol{U \Lambda U^{-1}} = \boldsymbol{U \Lambda V}$

Where $\boldsymbol{V} = \boldsymbol{U}^{-1}$

By multiplying $\boldsymbol{V}$ on the left of $\boldsymbol{P} = \boldsymbol{U \Lambda U^{-1}} = \boldsymbol{U \Lambda V}$

we have $\boldsymbol{VP} = \boldsymbol{\Lambda V}$

- Defining a set of row vectors $\boldsymbol{v}_i^T$ by **V**= $\begin{bmatrix} \boldsymbol{v}_0^T \\ \boldsymbol{v}_1^T \\ \dots \\ \boldsymbol{v}_{M-1}^T \end{bmatrix}$

- we find from $\boldsymbol{VP} = \boldsymbol{\Lambda V}$ $\quad \boldsymbol{v}_i^\top \boldsymbol{P} = \lambda_i \boldsymbol{v}_i^\top, i \in \mathcal{S}.$

- Therefore, $\boldsymbol{v}_i^T$ is the **left-eigenvector** associated with the eigenvalue λ_i.

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method

- We have $\boldsymbol{VU} = \boldsymbol{I}$, which implies that $\boldsymbol{v}_i$ and $\boldsymbol{u}_j$ are **bi-orthonormal**; i.e.,

$$\boldsymbol{v}_i^\top \boldsymbol{u}_j = \delta_{ij}, i, j \in \mathcal{S}.$$

- And $\boldsymbol{P}^2 = \boldsymbol{U\Lambda U}^{-1}\boldsymbol{U\Lambda U}^{-1} = \boldsymbol{U}\,\boldsymbol{\Lambda}^2\boldsymbol{U}^{-1}$
- By repeating the same procedure $(n - 1)$ times, we have

$$P^n = U\Lambda^n U^{-1} = \sum_{i\in\mathcal{S}} \lambda_i^n \boldsymbol{u}_i \boldsymbol{v}_i^\top = \sum_{i\in\mathcal{S}} \lambda_i^n E_i,$$

- where the matrices

$$E_i = \boldsymbol{u}_i \boldsymbol{v}_i^\top, i \in \mathcal{S},$$

are the **projection matrices**

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method

- By taking the transpose of ***VU* = *UV*= *I*** and expanding

$$I = (UV)^{\top} = \sum_{i \in S} v_i u_i^{\top} = \sum_{i \in S} E_i^{\top},$$

- We find

$$\sum_{i \in S} E_i = \sum_{i \in S} E_i^{\top} = I,$$

- which corresponds to the case $n = 0$ in the expansion formula

$$P^n = U \Lambda^n U^{-1} = \sum_{i \in S} \lambda_i^n u_i v_i^{\top} = \sum_{i \in S} \lambda_i^n E_i,$$

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method

- we may write the *n*-step transition probability from state *i* to state *j* as

$$P_{ij}^{(n)} = \sum_{k \in \mathcal{S}} \lambda_k^n u_{ki} v_{kj}, i, j \in \mathcal{S}, n = 0, 1, 2, \ldots.$$

- The state probability vector at step *n* is given from
- as

$$p^\top(n) = p^\top(0) P^n,$$

$$p^\top(n) = p^\top(0) P^n = \sum_{k \in \mathcal{S}} \lambda_k^n p^\top(0) u_k v_k^\top.$$

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method

- Thus, the probability that the Markov chain is in state *i* at time *n* is given by

$$p_i(n) = \sum_{k \in \mathcal{S}} \lambda_k^n \left(p^\top(0) u_k \right) v_{ki}, i \in \mathcal{S}.$$

- The marginal PMF $\boldsymbol{p(n)} = (p_0(n) \quad p_1(n) \ldots \quad p_i(n) \ldots p_{|\mathcal{S}|}(n))$

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method -example

- Assume

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

- The characteristic equation that determines the eigenvalues is given by

$$\det |P - \lambda I| = 0,$$

- And rearranged as

$$\det |I - \lambda^{-1} P| = \Delta(\lambda^{-1}) = 0,$$

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method -example

- where $\Delta(z)$ was defined earlier.
- Since we find that there are three roots for $\Delta(z) = 0$,
- $z_0 = 1$, $z_1 = -2$, and $z_2 = 4$, we readily find the three eigenvalues of
- ***P***: $\lambda_0 = z_0^{-1} = 1$, $\lambda_1 = z_1^{-1} = -1/2$, $\lambda_2 = z_2^{-1} = 1/4$
- And the corresponding right-eigenvectors are

$$u_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, u_1 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \text{and } u_2 = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}.$$

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method -example

- Thus we find

$$U = [u_0 u_1 u_2] = \begin{bmatrix} 1 & 4 & 4 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

- And its inverse

$$V = U^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 4 & 4 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \frac{1}{9}U,$$

- https://www.wolframalpha.com/calculators/eigenvalue-calculator

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method -example

- From which we find

$$V = U^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 4 & 4 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \frac{1}{9}U,$$

$$v_0^\top = \frac{1}{9}(1, 4, 4),\ v_1^\top = \frac{1}{9}(1, -2, 1),\ v_2^\top = \frac{1}{9}(1, 1, -2).$$

- The projection matrices are

$$u_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ u_1 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix},\ \text{and}\ \ u_2 = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & -2 \end{bmatrix}$$

$$E_0 = u_0 v_0^\top = \frac{1}{9}\begin{bmatrix} 1 & 4 & 4 \\ 1 & 4 & 4 \\ 1 & 4 & 4 \end{bmatrix},$$

$$E_1 = u_1 v_1^\top = \frac{1}{9}\begin{bmatrix} 4 & -8 & 4 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix},$$

$$E_2 = u_2 v_2^\top = \frac{1}{9}\begin{bmatrix} 4 & 4 & -8 \\ 1 & 1 & -2 \\ 2 & 2 & -4 \end{bmatrix}.$$

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method -example

- The n-step TPM is calculated as

$\lambda_0 = z_0^{-1} = 1$, $\lambda_1 = z_1^{-1} = -1/2$, $\lambda_2 = z_2^{-1} = 1/4$

$$E_0 = u_0 v_0^\top = \frac{1}{9}\begin{bmatrix} 1 & 4 & 4 \\ 1 & 4 & 4 \\ 1 & 4 & 4 \end{bmatrix},$$

$$E_1 = u_1 v_1^\top = \frac{1}{9}\begin{bmatrix} 4 & -8 & 4 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix},$$

$$E_2 = u_2 v_2^\top = \frac{1}{9}\begin{bmatrix} 4 & 4 & -8 \\ 1 & 1 & -2 \\ 2 & 2 & -4 \end{bmatrix}.$$

$$P^n = \sum_{i=0}^{2} \lambda_i^n E_i$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 4 & 4 \\ 1 & 4 & 4 \\ 1 & 4 & 4 \end{bmatrix} + \frac{1}{9}\left(-\frac{1}{2}\right)^n \begin{bmatrix} 4 & -8 & 4 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} + \frac{1}{9}\left(\frac{1}{4}\right)^n \begin{bmatrix} 4 & 4 & -8 \\ 1 & 1 & -2 \\ 2 & 2 & -4 \end{bmatrix}.$$

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method -example

- Assume the MC is initially at state 1;
- i.e., $\boldsymbol{p}^{\top}(0) = (1, 0, 0)$. Then,

$$\boldsymbol{P}^n = \sum_{i=0}^{2} \lambda_i^n \boldsymbol{E}_i$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 4 & 4 \\ 1 & 4 & 4 \\ 1 & 4 & 4 \end{bmatrix} + \frac{1}{9}\left(-\frac{1}{2}\right)^n \begin{bmatrix} 4 & -8 & 4 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} + \frac{1}{9}\left(\frac{1}{4}\right)^n \begin{bmatrix} 4 & 4 & -8 \\ 1 & 1 & -2 \\ 2 & 2 & -4 \end{bmatrix}.$$

$$\boldsymbol{p}^{\top}(n) = \boldsymbol{p}^{\top}(0)\boldsymbol{P}^n = \frac{1}{9}(1, 4, 4) + \frac{1}{9}\left(-\frac{1}{2}\right)^n (4, -8, 4) + \frac{1}{9}\left(\frac{1}{4}\right)^n (4, 4, -8)$$

$$= (p_0^{(n)} p_1^{(n)} p_2^{(n)}),$$

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method -example

- Assume the MC is initially at state 1;
- i.e., $\boldsymbol{p}^{\top}(0) = (1, 0, 0)$. Then,

$$\boldsymbol{p}^{\top}(n) = \boldsymbol{p}^{\top}(0)\boldsymbol{P}^{n} = \frac{1}{9}(1,4,4) + \frac{1}{9}\left(-\frac{1}{2}\right)^{n}(4,-8,4) + \frac{1}{9}\left(\frac{1}{4}\right)^{n}(4,4,-8)$$
$$= (p_0^{(n)} p_1^{(n)} p_2^{(n)}),$$

- where

$$p_0^{(n)} = \frac{1}{9} + \frac{4}{9}\left(-\frac{1}{2}\right)^{n} + \frac{4}{9}\left(\frac{1}{4}\right)^{n},$$
$$p_1^{(n)} = \frac{4}{9} - \frac{8}{9}\left(-\frac{1}{2}\right)^{n} + \frac{4}{9}\left(\frac{1}{4}\right)^{n},$$
$$p_2^{(n)} = \frac{4}{9} + \frac{4}{9}\left(-\frac{1}{2}\right)^{n} - \frac{8}{9}\left(\frac{1}{4}\right)^{n}.$$

Calculation of limiting Distribution (P^n): finite Chain- Spectral expansion method -example

- We can also verify that $p_0^{(n)} + p_1^{(n)} + p_2^{(n)} = 1$ for all n as expected, because the n-step TPM $\boldsymbol{P}^n$ is a stochastic matrix.
- We also find that, in the limit $_{n \to \infty}$, $p_0^{(n)} \to 1/9$, $p_1^{(n)} \to 4/9$, $p_2^{(n)} \to 4/9$
- These steady-state probabilities are independent of the initial probability $\boldsymbol{p}(0)$.
- The left eigenvector $\boldsymbol{v}_0$, associated with the eigenvalue $\lambda_0 = 1$, determines the steady-state solution.